Engineering Notes

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Controllability and Observability of Large Flexible Spacecraft in Noncircular Orbits

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Introduction

THE influence of noncircular orbits on flexible spacecraft flight control has received little attention in the technical literature. Such problems arise in the case of gravity-gradient or spin-stabilized spacecraft, moving in noncircular orbits, with very flexible appendages that are not aligned with the spacecraft principal axes. For noncircular orbits, there are some special problems with disturbances such as gravity gradient, solar pressure, atmospheric drag, etc. The design and ultimate development of these spacecraft require extensive analytical and experimental studies of their dynamics and control. The most important and difficult problems are the analysis of the dynamic response, the controllability and observability of the system, and then the determination of stability characteristics.

Of special interest here is the determination of the controllability and observability of large spacecraft structure. The physical system undergoing analysis may be described by a nontopological tree configuration.[1] The total response of the dynamic system is then considered to be in a nominal trajectory with perturbed motion with respect to the nominal trajectory. Furthermore, if the system is reducible in the sense of Lyapunov's definition,[2] some Lyapunov transformation will bring the variational equation relative to the nominal trajectory to a simple form associated with a time-invariant matrix. The controllability, observability, and stability characteristics are then determined.

Governing Equation of Motion and Its Transformation

The equations of motion for a dynamical system which can be discretized in terms of the generalized coordinate q have the form $\dot{q}=Q(q,u,t)$, where u represents a set of control inputs. For an autonomous system, $\dot{q}=Q(q,u)$ and, taking the separate components, we get equations of the form

$$\dot{q}=Q(q_1,\ldots,q_n,u_1,\ldots,u_p) \tag{1}$$

Solving the problem, in the most complete sense, means determining the function Φ such that

$$q^0=\Phi(c_1,\ldots,c_n,d_1,\ldots,d_p) \tag{2}$$

where $c_1,\ldots,c_n$ and $d_1,\ldots,d_p$ are, respectively, the initial conditions of $q_1,\ldots,q_n$ and $u_1,\ldots,u_p$ at $t=t_0$, and q^0 is obviously a corresponding nominal trajectory. Since many problems are intractable, in the sense that we cannot determine the function Φ explicitly, it is important to consider the characteristics in the neighborhood of a known characteristic. The situation is that we know the value of q^0 and u^0 for $t\geq t_0$ for a particular value of the intial point c and d, but we do not know the functions Φ for any range of values c and d. We then attempt to approximately determine and finally control the chracteristic in the neighborhood of the known q^0 and u^0.

Let us denote the neighboring characteristic by $q^0+\xi$ and $u=u^0+u^1$. Thus the displacement ξ from the undisturbed characteristic is such that variables ξ satisfy the equation

$$\dot{\xi}=\psi(t) \tag{3}$$

where

$$\psi(t)=Q(q_1^0+\xi_1,\ldots,q_n^0+\xi_n,u_1^0+u_1^1,\ldots,u_p^0+u_p^1)-Q(q_1^0,\ldots,q_n^0,u_1^0,\ldots,u_p^0)$$

If we expand it by a Taylor series and retain only terms of the first order in ξ, we obtain the linear approximation

$$\dot{\xi}=\Gamma(t)\xi+Bu^1 \tag{4}$$

where

$$\Gamma=\frac{\partial Q_i}{\partial q_j}\bigg|_{q=q^0,\,u=u^0}(i,j=1,2,\ldots,n)$$

and

$$B=\frac{\partial Q_i}{\partial u_k}\bigg|_{q=q^0,u=u^0}(k=1,2,\ldots,p)$$

Among the systems of linear differential equations of the first order, the simplest and best known are those with constant coefficients. It is therefore of interest to study systems that can be carried by a transformation into systems with constant coefficients. Lyapunov has called such systems reducible with the Lyapunov transformation (see Ref. 2).

If the system is reducible, then some Lyapunov transformation

$$\xi=L(t)x \tag{5}$$

can carry the variational equation (4) into a system

$$\dot{x}=Jx+L^{-1}Bu^1 \tag{6}$$

where

$$J=L^{-1}\Gamma L-L^{-1}\dot{L} \tag{7}$$

is a constant Jordan matrix with real characteristic values. Note that every system with periodic coefficients is reducible.[2]

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Controllability and Observability for the Reducible System

In many problems of theoretical nature and also in practical instrumentation design, it is of great interest to determine the controllability and observability of a system if its time-varying dynamic equations are transformed into their Jordan form [Eq. (6)] associated with constant eigenvalues. Based on the properties of the Lyapunov transformation [Eq. (5)] and the simple Jordan form [Eq. (6)], the necessary and sufficient criterion of the controllability and observability for the reducible system will be shown.

We consider the following linear autonomous n-vector control system:

$$\dot{x}=Jx+\hat{B}(t)u^1 \qquad y=\hat{C}(t)x \tag{8}$$

where $\hat{B}(t)=L^{-1}(t)B(t)$ and $\hat{C}(t)=C(t)L(t)$, and $C(t)$ is an $m\times n$ matrix given by the observation equation

$$y=C(t)\xi \tag{9}$$

Silverman and Meadows[3] show that if L is nonsingular, the controllability and observability of the system are not lost by the change of state variables given by Eq. (5). The total controllability means that the state can be driven from any $x(t_0)$ to any $x(t_1)$ for any t_0 and any $t_1>t_0$. In duality, the total observability indicates that the state $x(t_0)$ can be determined from the knowledge of the output $y(t_1)$ for any t_0 and any $t_1>t_0$.

We define the controllability matrix[4] of system [Eq. (8)] by

$$Q_c=[P_0(t)\,\vdots\,P_1(t)\,\vdots\ldots\vdots\,P_{n-1}(t)] \tag{10}$$

where

$$P_{k+1}=\left(-J+I\frac{d}{dt}\right)P_k \qquad P_0(t)=\hat{B}(t)$$

and I is an identity matrix. The observability matrix[4] is defined similarly by

$$Q_0=[S_0(t)\,\vdots\,S_1(t)\,\vdots\ldots\vdots\,S_{n-1}(t)] \tag{11}$$

where

$$S_{k+1}=\left(J^T+I\frac{d}{dt}\right)S_k \qquad S_0(t)=\hat{C}^T(t)$$

Theorem 1 (Ref. 4): System [Eq. (8)] is totally controllable on (t_0,t_1) if and only if the controllability matrix Q_c has rank n everywhere dense in (t_0,t_1). In duality:

Theorem 2 (Ref. 4): System [Eq. (8)] is totally observable on (t_0,t_1) if and only if the observability matrix Q_0 has rank n everywhere dense in (t_0,t_1).

To this end, consider the linear time-varying system with $L^{-1}(t)$ being written as

$$L^{-1}(t)=\sum_{i=1}^{r}f_i(t)F_i \tag{12}$$

where F_i's are constant matrices and $f_i(t)$ are sets of scalar time functions extracted from elements of $L^{-1}(t)$. In addition, assume the influence matrix B in the system equations of motion [Eq. (4)] is time-invariant. It thus follows that

$$e^{-Jt}L^{-1}(t)B=\left[\sum_{\alpha=0}^{n-1}\rho_\alpha(t)J^\alpha\right]\left[\sum_{\beta=1}^{r}f_\beta(t)F_\beta\right]B$$
$$=\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\rho_\alpha(t)f_\beta(t)J^\alpha F_\beta B \tag{13}$$

where the $\rho_\alpha(t)$ are scalar-valued functions of t. If there is a number m such that $J^\alpha F_\beta b_\gamma$ ($\alpha=0,\ldots,n-1$; $\beta=1,\ldots,r$; $\gamma=1,\ldots,m$) exists with n independent vectors $\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_n$, where we denote B by

$$B=[b_1\,\vdots\,b_2\,\vdots\ldots\vdots\,b_m]$$

with b_i ($i=1,\ldots,m$) being column vectors. By recalling that $\hat{b}_i$ ($i=1,\ldots,n$) can be considered as a basis of dimension$n$ space, we therefore obtain

$$\rho_\alpha(t)f_\beta(t)J^\alpha F_\beta b_\gamma=\sum_i^n \alpha\beta\gamma h_i(t)\hat{b}_i \tag{14}$$

To determine the controllability of the system, it is first noted that

$$\theta^k(t)=\frac{d^k}{dt^k}[e^{-Jt}L^{-1}(t)B]=e^{-Jt}P_k(t) \tag{15}$$

as can be shown by simple induction. Since e^{-Jt} is nonsingular for all t, the rank of the controllability matrix Q_c is equal to that of $W(t)$, which is

$$W(t)=[\theta(t)\,\vdots\,\theta^1(t)\,\vdots\ldots\vdots\,\theta^{n-1}(t)]$$

The controllability of the system can thus be determined by the rank test of the matrix $W(t)$. By using the elementary transformations which do not alter the rank of a matrix, we obtain

$$\begin{aligned}
&\text{rank}\Bigg[\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\rho_\alpha(t)f_\beta(t)J^\alpha F_\beta B,\\
&\quad\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}J^\alpha F_\beta B\frac{d}{dt}[\rho_\alpha(t)f_\beta(t)]\\
&\quad,\ldots,\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}J^\alpha F_\beta B\frac{d^{n-1}}{dt^{n-1}}[\rho_\alpha(t)f_\beta(t)]\Bigg]\\
&\geq\text{rank}\Bigg[\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{m}\rho_\alpha(t)f_\beta(t)J^\alpha F_\beta b_\gamma,\\
&\quad\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{m}J^\alpha F_\beta b_\gamma\frac{d}{dt}[\rho_\alpha(t)f_\beta(t)]\\
&\quad,\ldots,\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{m}J^\alpha F_\beta b_\gamma\frac{d^{n-1}}{dt^{n-1}}[\rho_\alpha(t)f_\beta(t)]\Bigg]\\
&=\text{rank}\left[\sum_{i=1}^{n}\hat{h}_i(t)\hat{b}_i,\ \sum_{i=1}^{n}\left(\frac{d}{dt}\hat{h}_i\right)\hat{b}_i,\ldots,\sum_{i=1}^{n}\left(\frac{d^{n-1}}{dt^{n-1}}\hat{h}_i\right)\hat{b}_i\right]
\end{aligned} \tag{16}$$

where

$$\hat{h}_i=\sum_{\alpha=0}^{n-1}\sum_{\beta=1}^{r}\sum_{\gamma=1}^{m}\alpha\beta\gamma h_i$$

With this background, two theorems can be developed as follows:

Theorem 3: System [Eq. (8)] is totally controllable only if $J^\alpha F_\beta B$ ($\alpha=0,\ldots,n-1$; $\beta=1,\ldots,r$) has rank n and if $\hat{h}_i(t)$ ($i=1,\ldots,n$) are linearly independent.

Proof: By applying the elementary transformations which preserve the rank of a matrix and noting Eq. (16), we obtain

$$\text{rank}[F_1B,\ldots,F_rB,JF_1B,\ldots,JF_rB,\ldots,J^{n-1}F_1B,\ldots,$$
$$J^{n-1}F_rB]\geq\text{rank}[W(t)]\geq$$

$$\text{rank}\left[\sum_{i=1}^{n} \hat{h}_i(t)\hat{b}_i, \sum_{i=1}^{n} \frac{d}{dt}\hat{h}_i(t)\hat{b}_i, \ldots, \sum_{i=1}^{n} \frac{d^{n-1}}{dt^{n-1}}\hat{h}_i\hat{b}_i\right]$$

If $\hat{h}_i (i=1,\ldots,n)$ are linearly independent, it is easy to show that

$$\text{rank}\left[\sum_{i=1}^{n} \hat{h}_i(t)\hat{b}_i, \sum_{i=1}^{n} \frac{d}{dt}\hat{h}_i(t)\hat{b}_i, \ldots, \sum_{i=1}^{n} \frac{d^{n-1}}{dt^{n-1}}\hat{h}_i\hat{b}_i\right]$$

$$=\text{rank}[\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n] = n$$

This completes the proof.

In duality, a theorem of observability for the reducible system can be established as follows:

Theorem 4: System [Eq. (8)] is totally observable only if $(CF_\beta J^\alpha)^T$ $(\alpha=0,\ldots,n-1, \beta=1,\ldots,r)$ has rank n and if its corresponding functions $\hat{h}_i(t)$ $(i=1,\ldots,n)$ are linearly independent.

The interpretation of this result is that a large number of time-variant system modes can be controlled by a few actuators and sensors. In fact, the number of actuators (or sensors) is determined by the number of columns of the influence matrix B (or rows of the matrix C); however, it is still required that control devices be located so as to satisfy the rank conditions and function independence tests of the theorems. Some trial and error of locations may be necessary subject to physical constraints, as well as the economic concern.

Discussion and Conclusion

A large majority of large flexible dynamic systems are nonlinear over a wide range of amplitudes of the dynamical quantities involved. Even though these systems contain nonlinearities in their normal range of performance, a linearized form of their state equation will be a valid approximation, provided that the state variables involved do not vary too widely from their nominal states about which linearization takes place. Moreover, the results obtained from such a linear model should give a qualitative idea of the way in which system performance will differ from model behavior. For these reasons, we introduce in this paper a linear model for a large flexible structure.

If the linear system is reducible in the sense of Lyapunov, the time-variant characteristic matrix Γ can be transformed to a time-invariant Jordan matrix. Our new results show that controllability and observability of a large flexible spacecraft can be determined from that simple Jordan form and the properties of the Lyapunov transformation.

It is clear that the eigenvalues of Γ, when Γ is constant, play a very similar part in the solution of the variational equation to that played by the characteristic exponent when Γ is reducible. It is natural, therefore, to extend the use of the term "characteristic exponents" so as to include the case when the elements of Γ are constant. In a problem in which Γ is a constant matrix, the exponents are its eigenvalues. Consequently, this may initiate the idea of updating or adapting some existing control theory from the time-invariant system to the reducible dynamic system.

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Evaluation of Mass Properties by Finite Elements

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Introduction

When determining the dynamic behavior of a space body a fundamental requirement is the accurate calculation of the body's mass properties. The usual method for performing this operation is to view the body as an assemblage of small elements for which standard formulas exist, evaluate the mass properties at the element level from these formulas and recover in total the properties of the body as a whole. As the shape of the body becomes more complicated the accuracy of this calculation may diminish. The method presented herein follows this same basic philosophy differing only in the calculation at the element level, wherein a finite element type analysis is employed.

The application of isoparametric finite elements for the approximation of boundary value problems defined over irregular domains is a standard procedure. Unlike this usual application for finite elements the method presented in this note does not seek the solution of a boundary-value problem but rather the evaluation of some simple integrals defined over the domain.

Analysis

Let Ω be the domain and x_i, $i=1,2,3$ be a global coordinate system with basis vectors $e_i, i=1,2,3$. The particular properties of interest are the mass M, the first mass moment Q and the moment of inertia I. When expanded against the basis e_i these properties take the form

$$M=\int_\Omega \rho \, dx \tag{1a}$$

$$Q=Q_i e_i \qquad Q_i=\int_\Omega \rho x_i \, dx \tag{1b}$$

$$I=I_{ij} e_i \otimes e_j \qquad I_{ij}=\int_\Omega \rho(\delta_{ij} x_k x_k - x_i x_j)\, dx \tag{1c}$$

where ρ is the density of the body and δ_{ij} is the Kronecker delta; in addition, the usual Cartesian summation convention has been employed.

The domain Ω can be viewed as an assemblage of E finite elements $\{\Omega_e\}_{e=1}^{E}$, two such elements being shown in Fig. 1. The number of nodes on element e is denoted as N_e; for the elements shown in Fig. 1 N_e is 8 and 20, respectively. The location of an arbitrary point x in Ω_e can be interpolated with respect to the known locations of the nodes denoted as x_i^α, $\alpha=1,\ldots N_e$. This interpolation takes the form

$$x_i=\sum_{\alpha=1}^{N_e} N_\alpha(\xi) x_i^\alpha \tag{2}$$

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